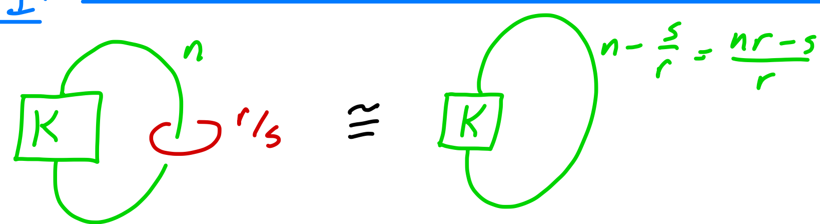
V More on Dehn Surgery

A. Altering Surgery Diagrams

we would like to know how to manipulate Dehn surgery descriptions of 3-manifolds

lemma 1:

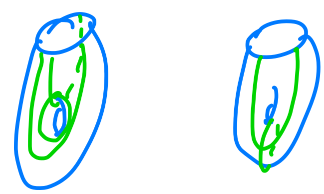


$n, r, s \in \mathbb{Z}$

called a slam dunk

Proof: doing n-surgery on K is the result of removing a nbhd of K from S^3 and gluing in $S^1 \times D^2$ to ∂S^3_K by

$$\phi = \begin{bmatrix} 0 & 1 \\ -1 & n \end{bmatrix}$$



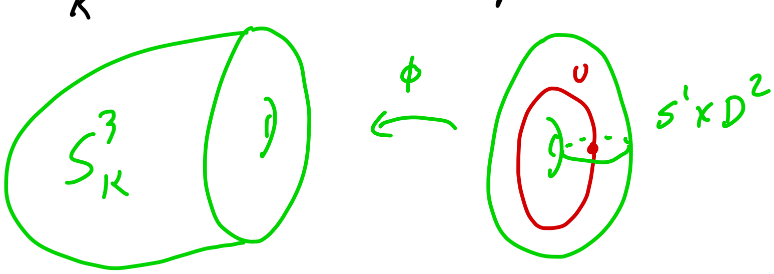
so $S^3_K(n) = S^3_K \cup_\phi S^1 \times D^2$

the unknot U in picture is a meridian to K so isotop U to ∂S^3_K and then transfer to $S^1 \times D^2$ via ϕ^{-1}

$$\phi^{-1}(U) = \begin{pmatrix} n & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

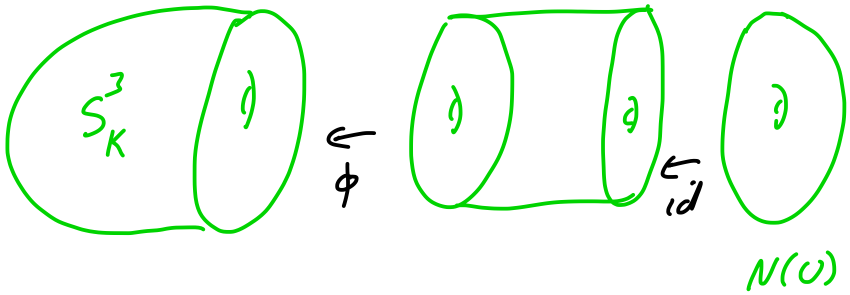
U $\quad$ $S^1 \times \{pt\}$

so in $S^3_K(n)$ U is isotopic to



now a nbhd $N(U)$ of U is a subset of $S^1 \times D^2$ such that $\overline{S^1 \times D^2 - N(U)} \cong T^2 \times [0,1]$

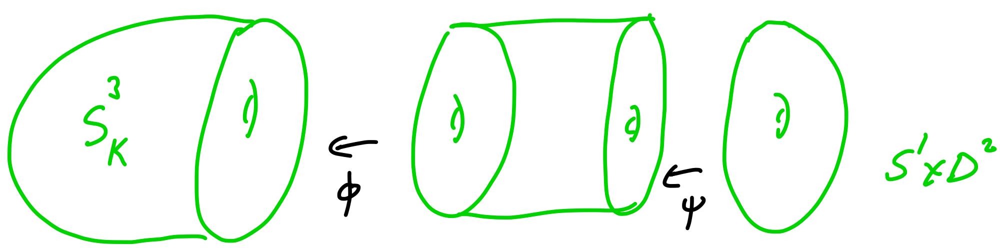
so $S^3_K(n)$ is



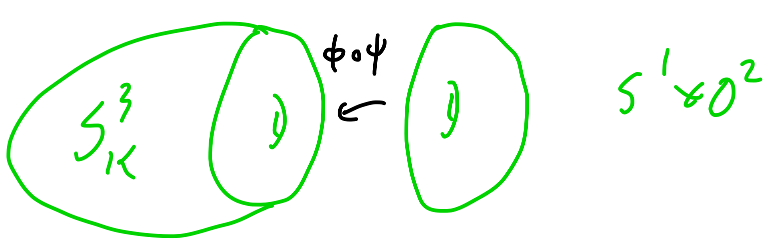
now for r/s surgery on U we remove $N(U)$ and glue by a map

$$\psi = \begin{pmatrix} s' & s \\ r' & r \end{pmatrix} \quad \text{s.t. } \det \psi = 1$$

so $(S^3_K(n))_U(r/s)$ is



by lemma I.7 this is the same as



$$\phi \circ \psi = \begin{pmatrix} 0 & 1 \\ -1 & n \end{pmatrix} \begin{pmatrix} s' & s \\ r' & r \end{pmatrix} = \begin{pmatrix} r' & r \\ -s' + nr' & -s + nr \end{pmatrix}$$

so meridian maps to $\begin{pmatrix} r \\ -s+nr \end{pmatrix}$

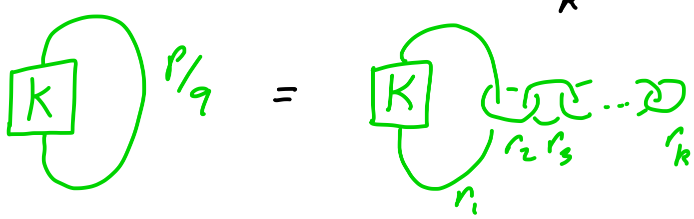
ie this is $n - s/r$ surgery on K

exercise: to express surgery on U we needed a prefered framing on U, make sure the framing on U when pushed into $S^1 \times D^2$ was unchanged, i.e. ϕ^{-1} didn't change it! (necessary for our description of ϕ)

Corollary 2:

given p, q relatively prime, one can find $r_1 \dots r_k$ such that $p/q = r_1 - \cfrac{1}{r_2 - \cfrac{1}{\ddots - \cfrac{1}{r_k}}}$

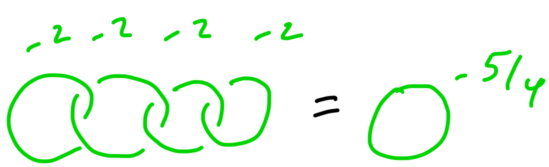
then



Proof: just slam dunk $k-1$ times

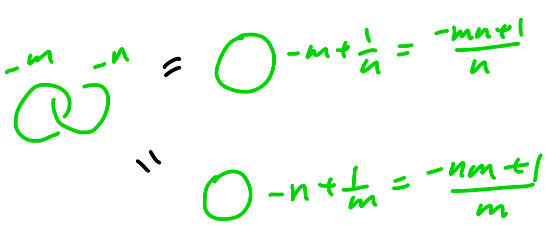
example:

1)



both describe $L(5,4)$

2)



so $L(mn-1, m) \cong L(mn-1, n)$

very cool, this is not completely obvious!

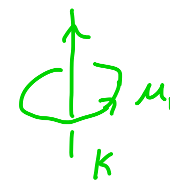
Remark: Cor 2 + Lickorish, Wallace Th^{m} (Th^{m} 1.5) says that any closed oriented 3-mfd is Dehn surgery on a link in S^3 with all surgery coefficients being integers
(actually a careful look at the proof of Th^{m} I.5 already shows this!)

For our next move we need linking numbers

if K_1 and K_2 are oriented knots in a homology sphere M

then $[K_2] \in H_1(M_{K_1}) \cong \mathbb{Z}$ gen by $[\mu_1]$

lemma III.14



so $[K_2] = m[\mu_1]$ some $m \in \mathbb{Z}$

we define the linking number to be $lk(K_1, K_2) = m$

recall $\exists$ an embedded surface $\Sigma_1 \subset M$ such that $\partial \Sigma_1 = K_1$ (as oriented manifolds)

note: $\Sigma_1 \cap \mu_1 = +1$

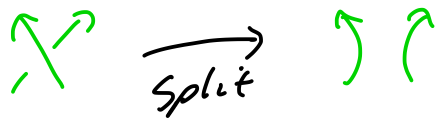
so $lk(K_1, K_2) = m(\Sigma_1 \cap [\mu_1]) = \Sigma_1 \cap (m[\mu_1])$
$= \Sigma_1 \cap [K_2]$

now let's compute $Lk(K_1, K_2)$ for $K_1, K_2 \subset \mathbb{R}^3 \subset S^3$

- project K_1 to xy-plane

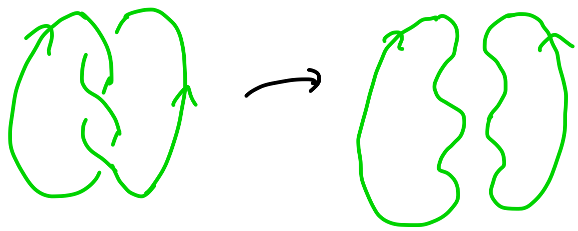
• construct a Seifert surface as follows

1) at each crossing of K_1

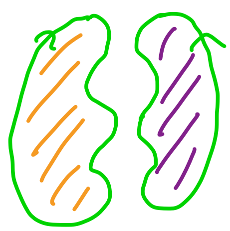


so that or$^{\underline{n}}$ is respected

get a bunch of circles

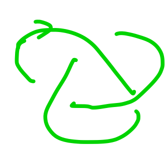


2) pick disks in $\mathbb{R}^2$ that these circles bound



3) at each crossing glue in a half twisted strip to creat a surface w/ $\partial = K_1$

this gives a surface in $\mathbb{R}^3$ that is almost in xy-plane and $\partial = K_1$

<u>exercise</u>: find surface for 

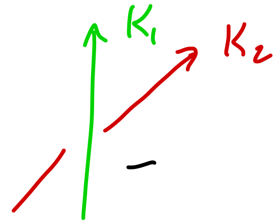
now to compute linking look at the diagram

1) think of trying to pull K_2 towards you

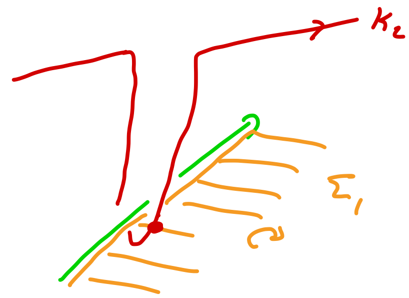
you only get stuck when K_2 passes under K_1

so the only place K_2 can intersect Σ_1 is near an under crossing

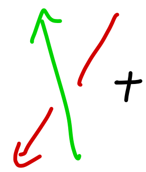
2) at an undercrossing you see



pull K_2 up to see



negative intersection point!

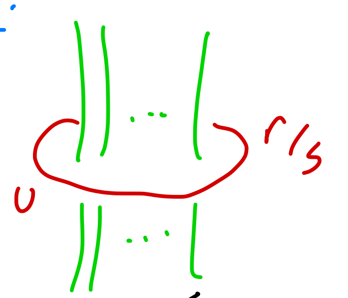


so $lk(K_1, K_2) = \sum\limits_{\substack{\text{crossings} \\ \text{of } K_2 \,\underline{\text{under}}\, K_1}} \varepsilon_c$ where ε_c is sign above

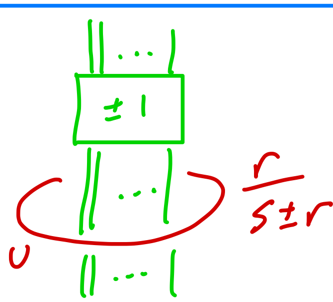
note: the Seifert longitude for K is exactly the curve λ with $lk(K, \lambda) = 0$

all other longitudes of the form $\lambda + m\mu$ where μ is the meridian of K and $m \in \mathbb{Z}$

lemma 3:



$\cong$



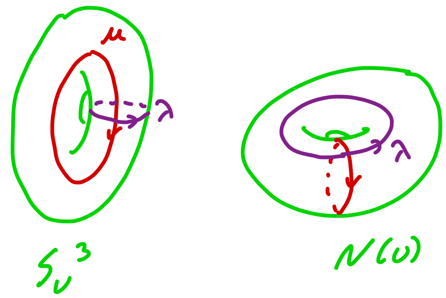
link with cpts $K_1 \ldots K_n$ and surgery coeff $r_1 \ldots r_n$

new surgery coeff on image of K_i is $r_i' = r_i \pm lk^2(K_i, U)$

(Rolfsen twist)

Proof: let U be the unknot

$S^3_U = S^1 \times D^2$



let $\Psi : S^3_U \to S^3_U$ be given by $(\phi, (r,\theta)) \mapsto (\phi, (r, \theta \pm \phi))$

on on $\partial(S^3_U)$, Ψ given by

$$\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$$

now $S^3_U(r/s) = S^3_U \cup_f S^1 \times D^2$ where $f = \begin{pmatrix} s' & s \\ r' & r \end{pmatrix}$

we can build a diffeomorphism

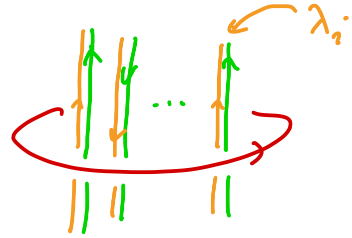
$$\begin{array}{ccccc} S^3_U(r/s) = & S^3_U & \cup_f & S^1 \times D^2 & \\ & \downarrow \Psi & & \downarrow id & \\ & S^3_U & \cup_{\Psi \circ f} & S^1 \times D^2 & = S^3_U\left(\frac{r}{s \pm r}\right) \end{array}$$

$$\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} s' & s \\ r' & r \end{pmatrix} = \begin{pmatrix} s' \pm r' & s \pm r \\ r' & r \end{pmatrix}$$

so lemma is clear except for surgery coefficients r_i'

to sort this out, let's see how the longitude and meridian, λ_1, μ_1, of K_i change under Ψ

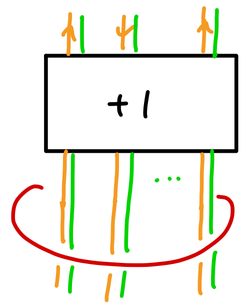
near U we have (only focus on K_1, ignore others)



suppos k upstrands and $n-k$ down

i.e. $lk(U, K_i) = 2k - n$

when we do a + twist we get



each orange strand goes under each green strand one time

if arrows agree   if not

so each of the k upstrands contributes

$$lk(U, K_i)$$

and each of the $(n-k)$ down strands contrib.

$$-lk(U, K_i)$$

$\therefore$ linking of orange and green is

$$(lk(U, K_i))^2$$

$$\therefore \psi(\lambda_i) = \lambda_i' + (lk(U, K_i))^2 \mu_i'$$

$$\psi(\mu_i) = \mu_i'$$

where λ_i', μ_i' are long/merid of image of K_i

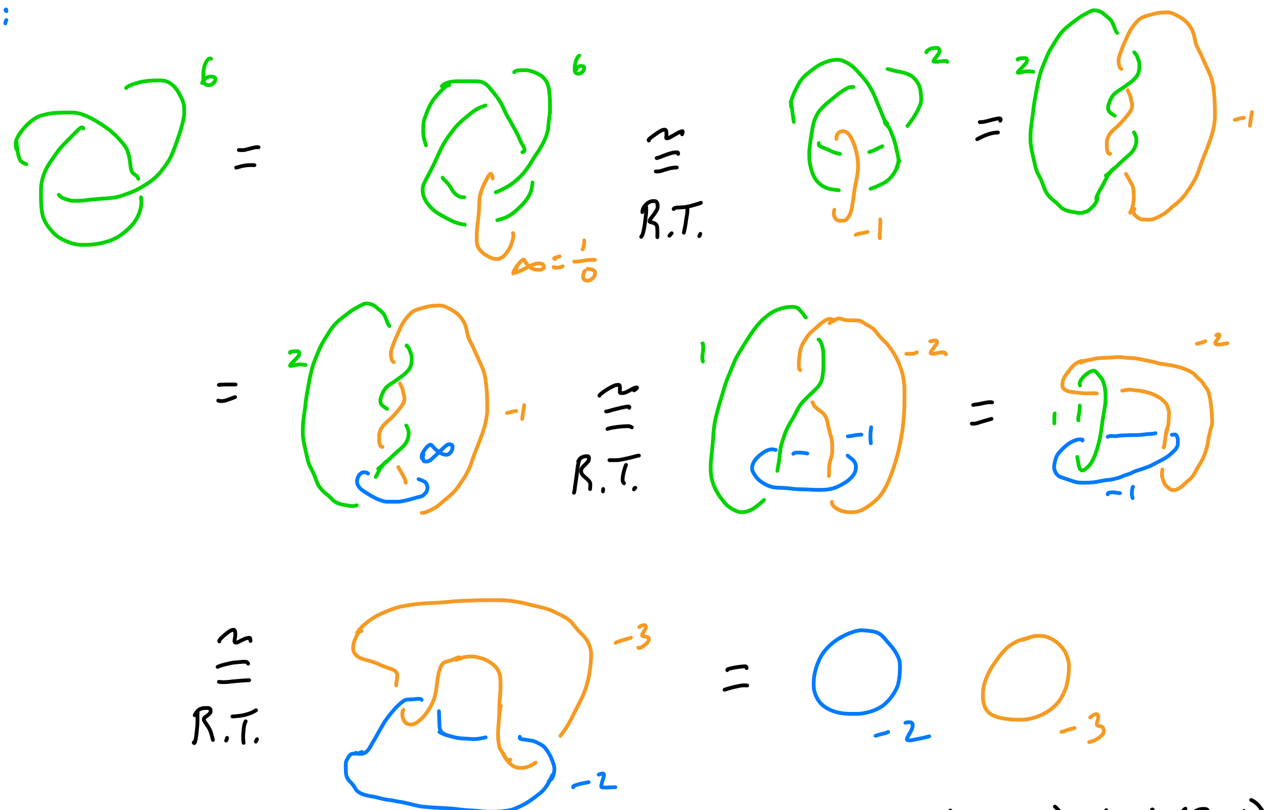
$\therefore r_i = p_i/q_i$ in (λ_i, μ_i) coords goes to

$$\begin{pmatrix}1 & 0\\ (lk(U,K_2))^2 & 1\end{pmatrix}\begin{pmatrix}q_2'\\ p_2'\end{pmatrix} = \begin{pmatrix}q_2'\\ p_2 + q_2\cdot lk(U,K_2)^2\end{pmatrix}$$

in λ_2', μ_2' coords so surgery coeff

$$r_2' = \frac{p_2'}{q_2'} + (lk(U,K_2))^2$$

Example:



$\cong L(2,1) \# L(3,1)$

exercise:

if M = Dehn surgery on L,

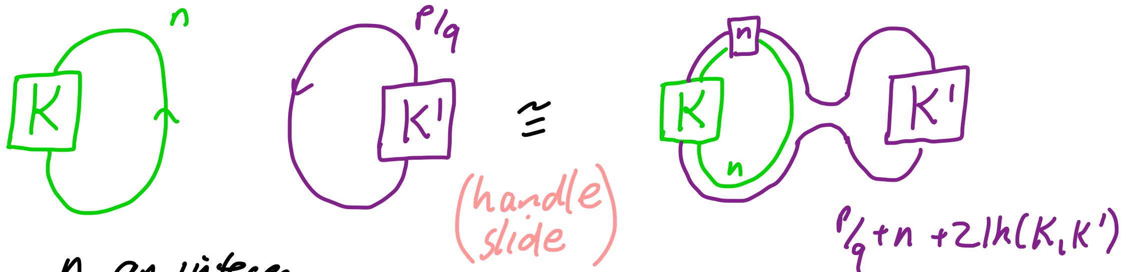
M' = " " L'

then $M \# M'$ = Dehn surgery on $L \cup L'$

L and L' separated by an $\mathbb{R}^2$

later we will see it is somewhat unusual to get a connected sum by surgery on a knot

lemma 4:



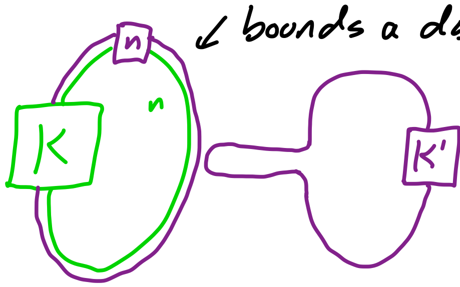
n an integer

K and K' can link

if one arrow reversed, the new surgery coeff is
$p/q + n - 2\,lk(K, K')$

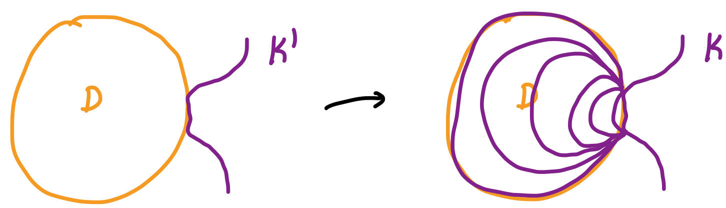
Proof:

↙ bounds a disk D in $S^3_n(K)$

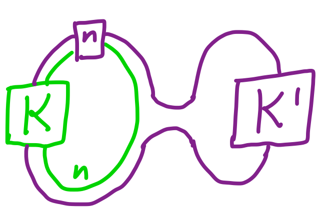


push a point on K' near ∂D

now use D to guide an isotopy



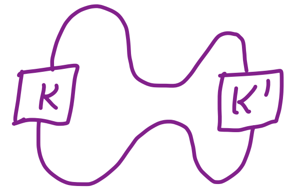
so in $S^3_K(n)$, K' is isotopic to



we now need to see what the surgery coeff.

becomes after this isotopy

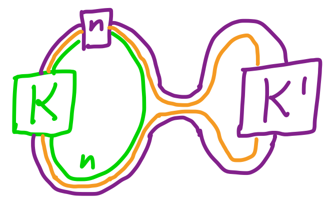
call



K''

under the above isotopy the longitude λ' of K'

goes to



(just push it over D as well)

let $\lambda = \lambda_K + n\mu_{K'} + \lambda_{K'}$

<u>note</u>: $\mu_{K'}$ for K' is still a meridian after isotopy

so in $\lambda, \mu_{K'}$ coords the surgery coeff. on K'' is p/q (exercise, if not clear!)

but λ is <u>not</u> the longitude of K''

let's compute the linking between λ and K''

the linking comes from undercrossings of λ under K'', they are of 2 types

1) undercrossings w/ K'
2) " " w/ K

type 1) crossings are of 2 types

a) a strand of λ parallel to K goes under K'

b) a strand of λ parallel to K' goes under K'

type a) crossings contribute 0 to lk since $lk(\lambda_{K'}, K') = 0$

type b) crossings contribute $lk(K, K')$

similarly, type 2) crossings also contribute $lk(K, K') + n$ to the linking

so $\lambda_{K''} = \lambda - (2lk(K, K') + n)\mu_{K''}$

now in $\lambda_{K''}, \mu_{K''}$ words

$$p\mu_{K''} + q\lambda = p\mu_{K''} + q(\lambda_{K''} + (2lk(K,K') + n)\mu_{K''})$$
$$= (p + q(n + 2lk(K,K')))\mu_{K''} + q\lambda_{K''}$$

so surgery coeff is $p/q + n + 2lk(K, K')$ ▦

a <u>blow up</u> of a surgery diagram is the addition of an unknotted componed, unlinked from the rest of the diagram, with surgery coeff. ± 1

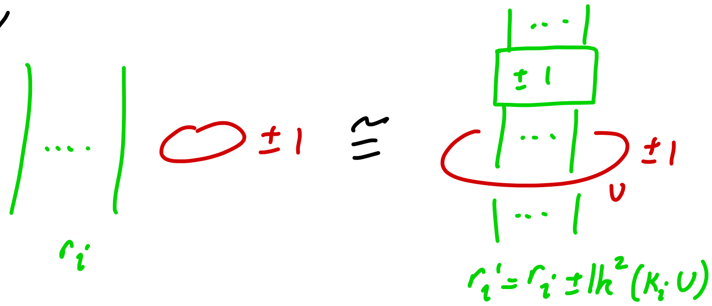
a <u>blow down</u> is the removal of such a component

note:

so blowing up and down do not affect the manifold described by the diagram!

exercise:

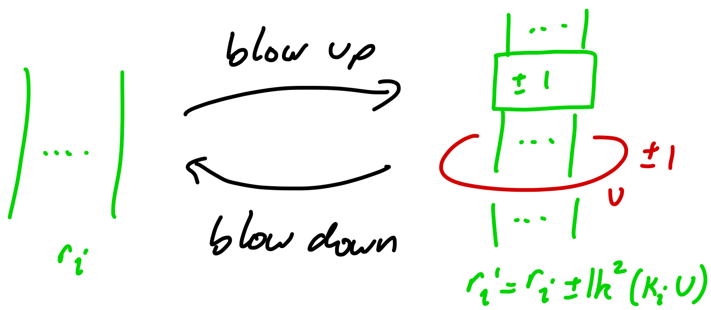
show



by using handle slides

(could also use Rolfsen twist)

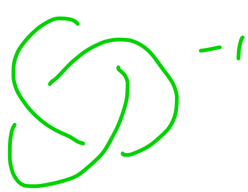
so some times people define



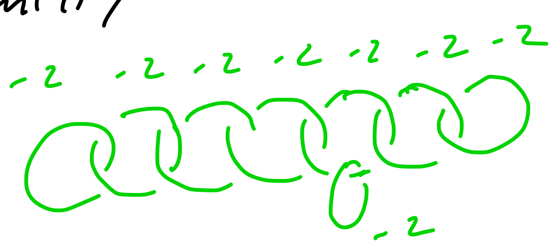
example:

recall we said earlier the Poincaré homology sphere

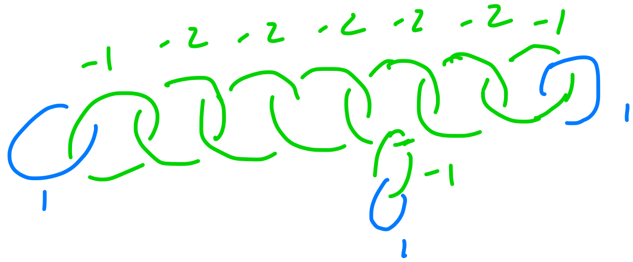
is



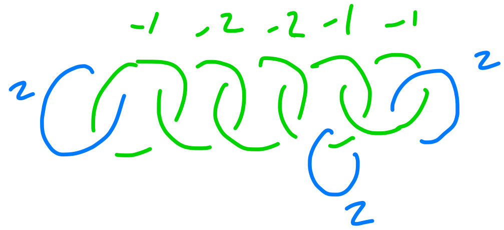
let's identify



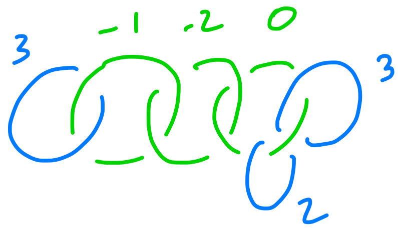
blow up at right left and bottom



blow down all -1 unknots

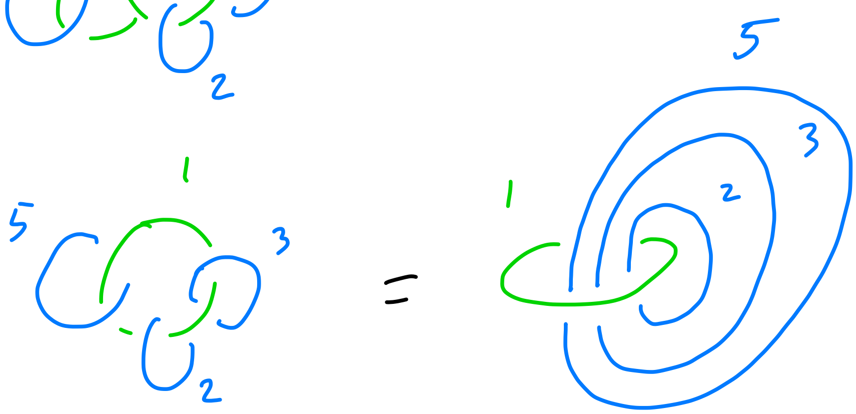


blow down left and right -1 unknots

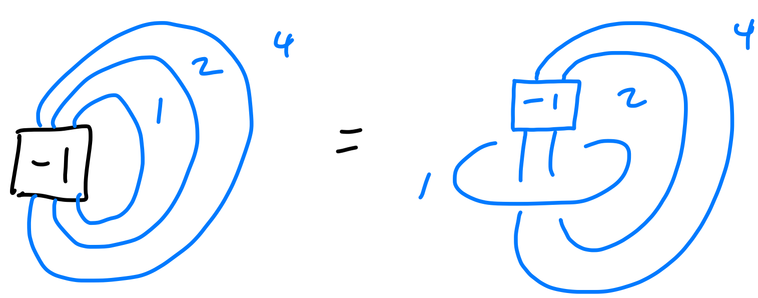


blowdown the -1 unknot

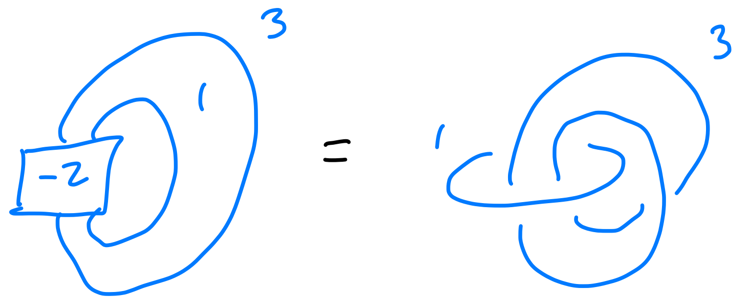
repeat



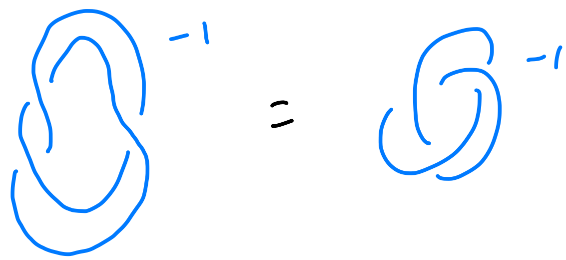
blow down +1 unknot



repeat



repeat

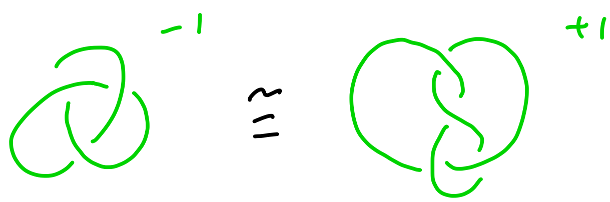


exercise:

1) show

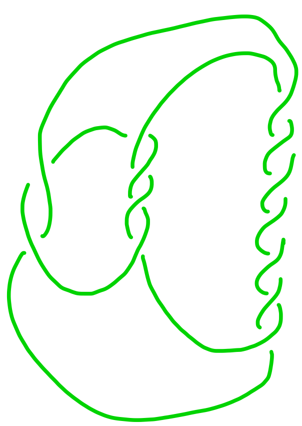
is the Poincaré homology sphere

2) show



$\pm L(18,5)$

3) K $=$ $P(-2,3,7)$



show $M_K(18) \cong L(18,13)$
(or $L(18,5) = -L(18,13)$)

and

$M_K(19) \cong \pm L(19,12)$
(or $L(19,7) = -L(19,12)$)

(Hard!)

Thm (Kirby, 1978):

two surgery diagrams in S^3 with integral surgery coefficients are diffeomorphic

$$\iff$$

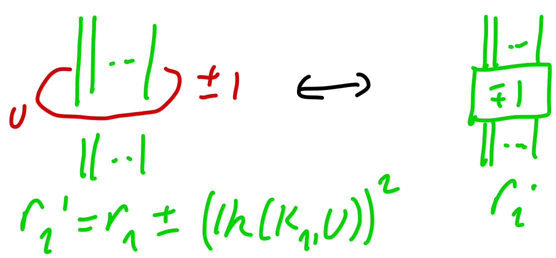
they are related by a sequence of blow ups/downs and handle slides

Moreover, any orientation preserving diffeom. can be realized this way

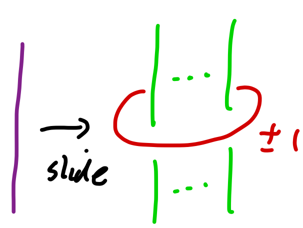
we prove this in the next section, but for now we use it to prove

Thm 5 (Fenn-Rourke, 1979):

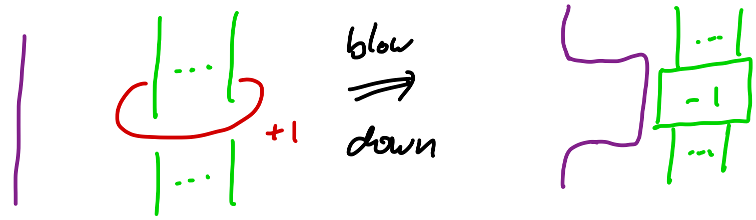
Kirby's theorem is true without handle slides! (only need blowups and blowdowns)

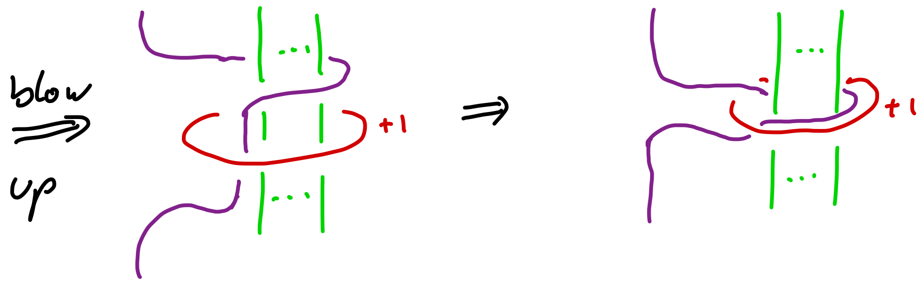


Proof: first note we can do the following handle slide via blowups & blowdowns



indeed



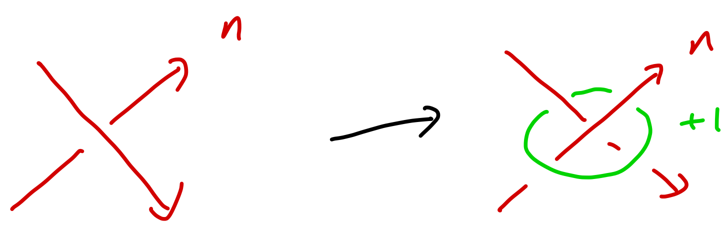


note: this works even if purple curve runs through red curve in any way

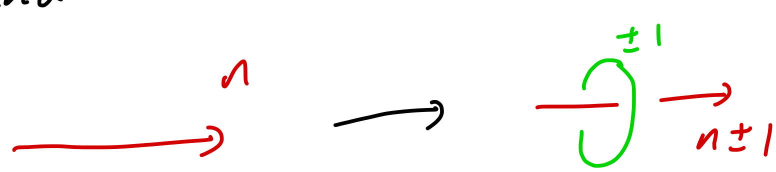
exercise: show framings on curves work out correctly

of course we could deal with -1 framed case in same way

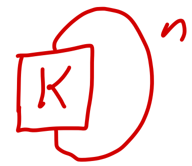
now note



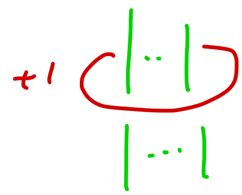
and



so by blowups we can turn any knot



into



now can slide of this unknot as above with blowup/downs and then blow down all the green to get back to K with something slid over it!

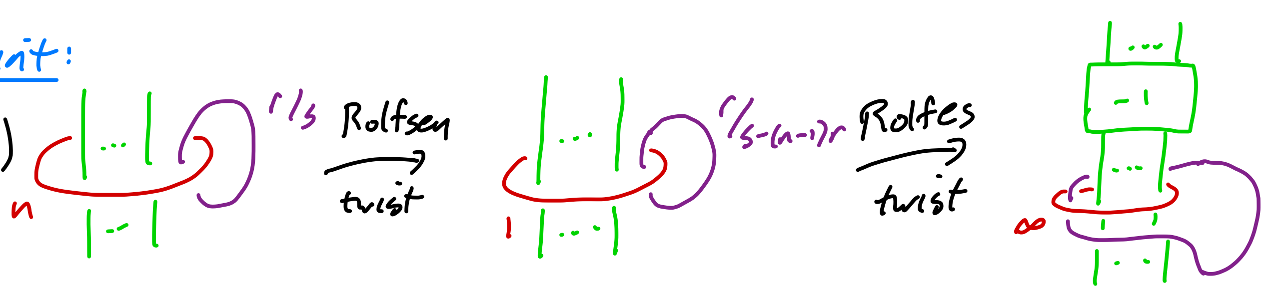
Corollary 6:

Surgery diagrams in S^3 with rational coeff. are diffeomorphic $\Longleftrightarrow$ they are related by Rolsen twists

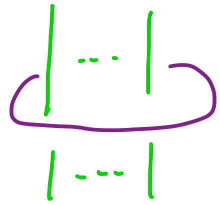
first we need

exercise: Show a slam dunk can be done by Rolfsen twists

Hint:

1) 

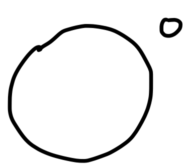
$$\frac{r}{s-(n-1)r} - 1 = \frac{nr-s}{s-(n-1)r}$$

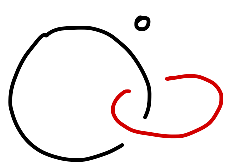
$\xrightarrow[\text{twist}]{\text{Rolfsen}}$  $\dfrac{nr-s}{s-(n-1)r+nr-s} = n - s/r$

2) note blowup/down are Rolfsen twist now use trick in Thm 5 to do general case.

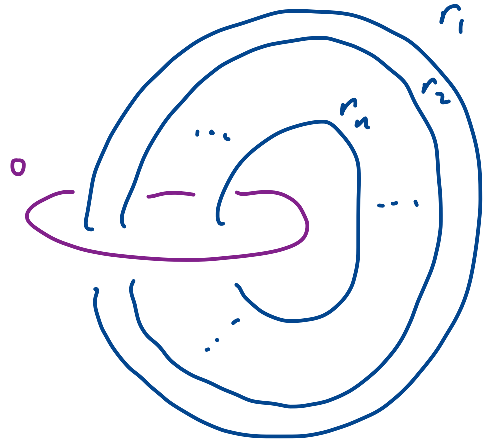
Proof: first use slam dunks (which are Rolfsen twists) to write surgery diagrams with integer coeff.
now they are related by blowup/downs by thm 5, but these are also Rolfsen twists!

B. Seifert Fiber Spaces

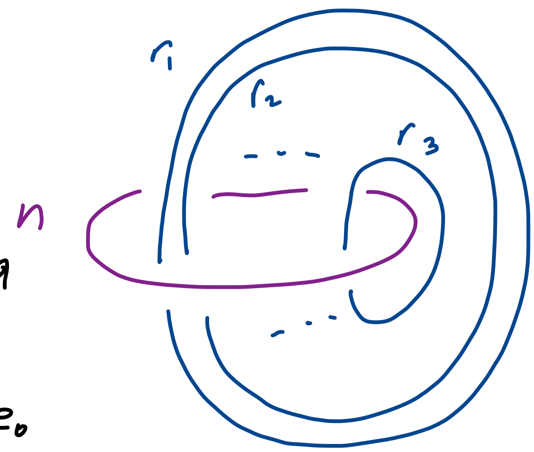
exercise:  is $S^1 \times S^2$

and  is $S^1 \times \{point\}$

from our discussion of Seifert fiber spaces (section IV.B) we see any SFS over S^2 can be written



by Rolfsen twists one can arrange all the $r_i < -1$ to get



usually denoted e_0

<u>exercise</u>: there is a unique way to do this

these are called the normalized Seifert invariants of the singular fibers

we denote the above SFS by

$$M(0, e_0; -\tfrac{1}{r_1}, \dots, -\tfrac{1}{r_n})$$

genus of base 0

$e = e_0 + \sum -\frac{1}{r_i}$ is called the rational <u>Euler number</u>

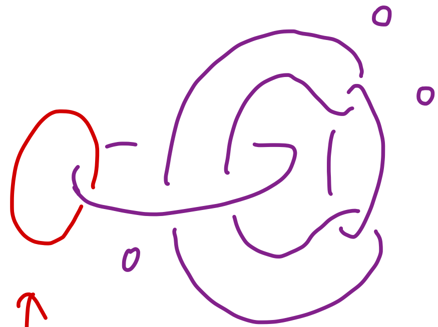
<u>exercise</u>:

1) $M(0, e_0; -\frac{1}{r_1}, \dots, -\frac{1}{r_n})$ has the same rational

homology as S^3 $\Leftrightarrow$ $e \neq 0$

2) $M(0, e_0; -\frac{1}{r_1}, \dots, -\frac{1}{r_n})$ has a horizontal incompressible surface $\Leftrightarrow$ $e = 0$

3) show

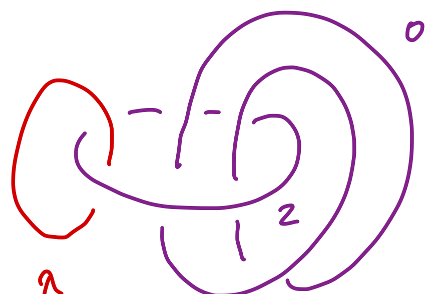


is T^3

$S^1 \times \{pt\} \subset S^1 \times T^2$

Hint: maybe later

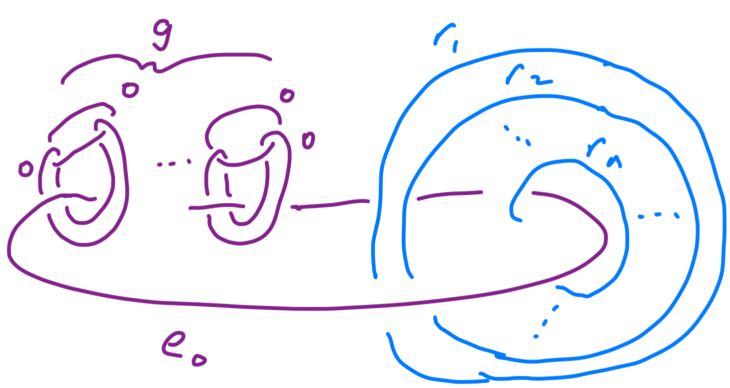
4) Show the orientable S^1-bundle over $\mathbb{R}P^2$ is



fiber

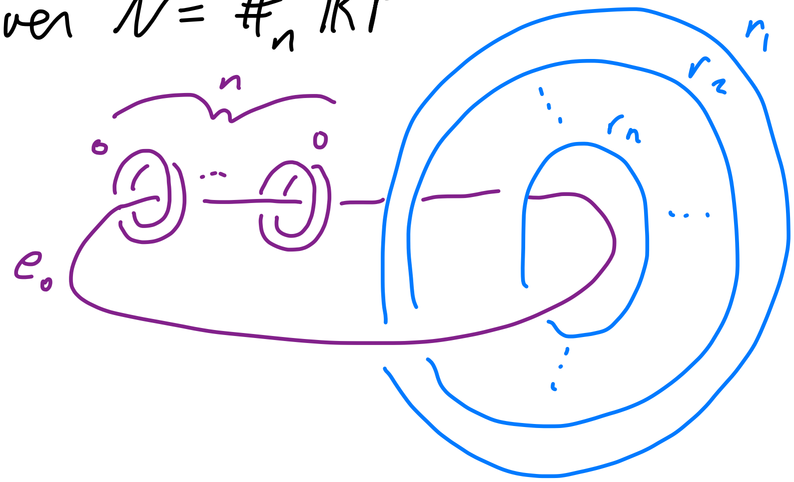
Hint: maybe later

from above not hard to show that a SFS over a surface of genus g with normalize Seifert invariants can be written



$$M(g, e_0; -\tfrac{1}{r_1}, \dots -\tfrac{1}{r_n})$$

and over $N = \#_n \mathbb{R}P^2$



$$M(-n, e_0; -\tfrac{1}{r_1}, \dots, -\tfrac{1}{r_n})$$

Th$^{\underline{m}}$ 7:

if $M = M(g, e_0; -\frac{1}{r_1}, \dots, -\frac{1}{r_n})$ and K is a regular fiber then a/b surgery on K is

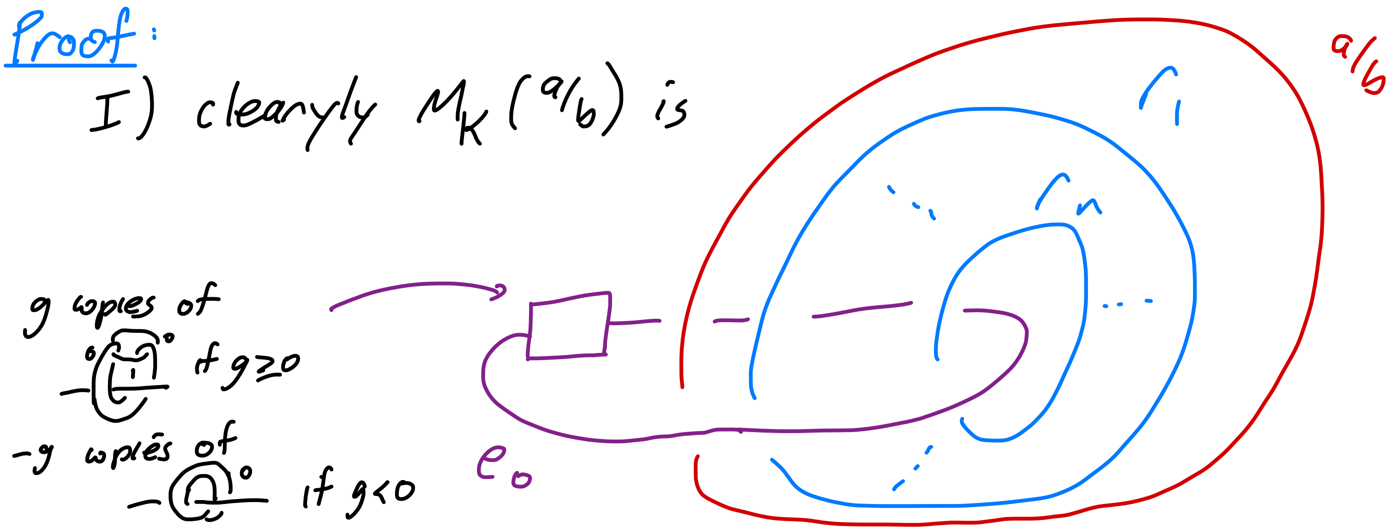
I) $M(g, e_0 - (n+1); -\frac{1}{r_1}, \dots, -\frac{1}{r_1}, \frac{(n+1)a - b}{a})$

if $a/b \neq 0$ and $b = na + r \quad 0 \leq r < a$

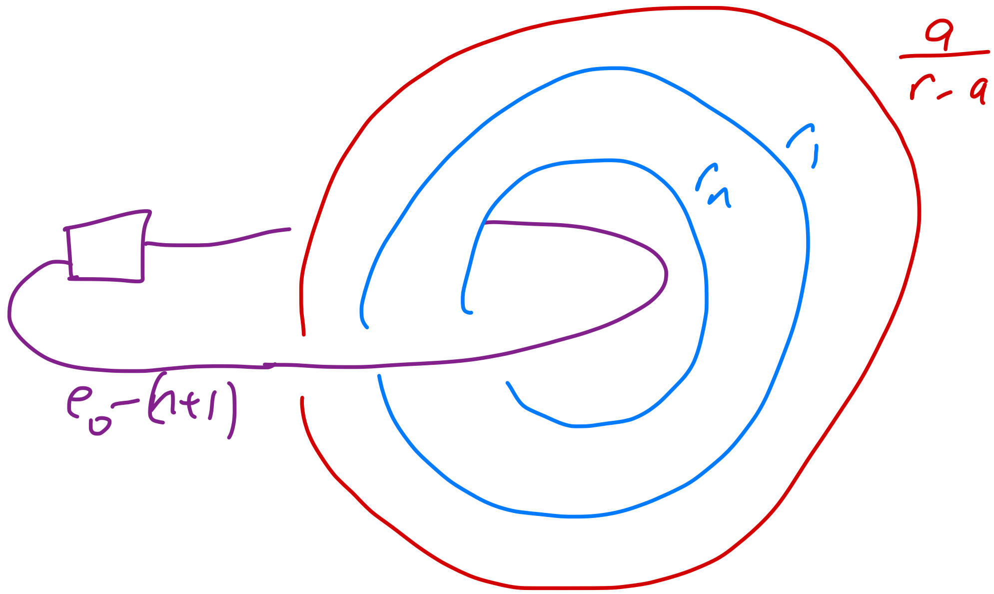
II) $\#_n L(a_i, b_1) \#_m S^1 \times S^2$ if $a/b = 0$ and $r_1 = -(a_i/b_1)$

Proof:

I) clearly $M_K(a/b)$ is



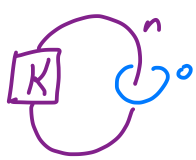
Rolfsen twist red curve $(n+1)$ times to get



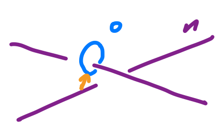
$$\text{so } M_K(a/b) = M\left(g, e_0-(n+1); \frac{-1}{r_1}, \dots, \frac{-1}{r_n}, \frac{a-r}{a}\right)$$

II) we need a lemma

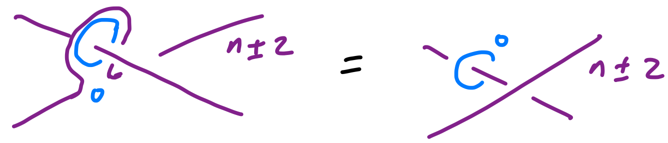
lemma 8:

> suppose 
>
> is part of a surgery diagram
>
> K can link other components but the meridian can't.
>
> then removing K and the meridian from the diagram gives the same 3-manifold

Proof: note at a crossing of K we can isotop meridian to see



do indicated handle slide to get

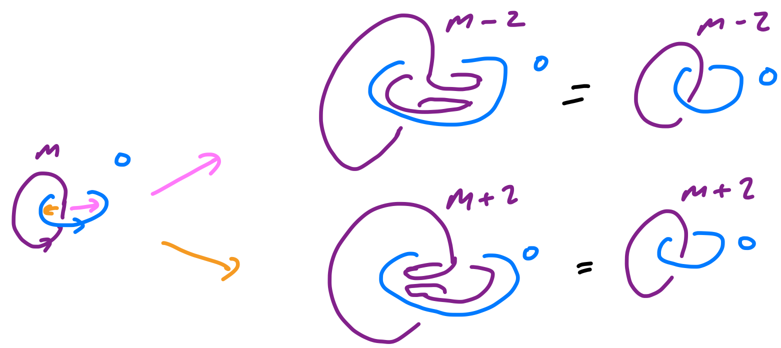


so we can unknot K by crossing changes

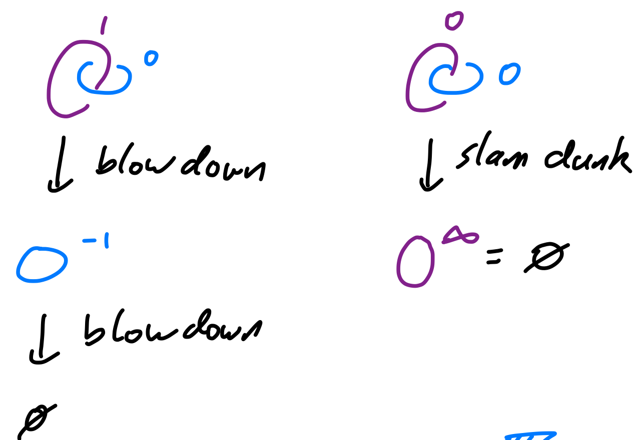
similarly, we can unlink K from rest of surgery diagram to get

$\cup$ rest of diagram

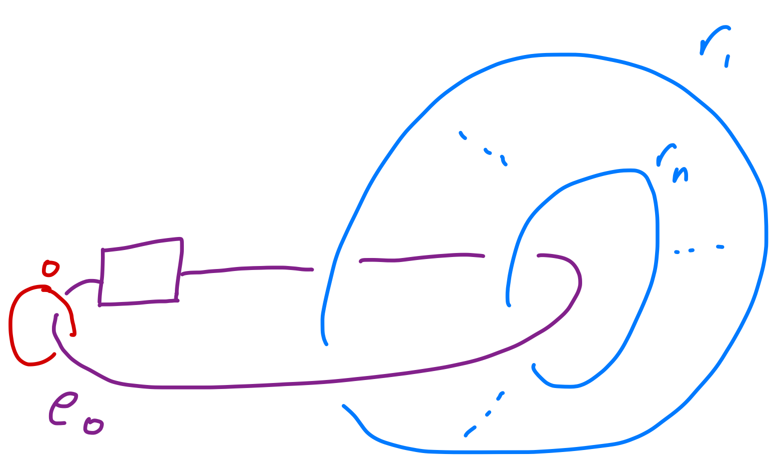
but



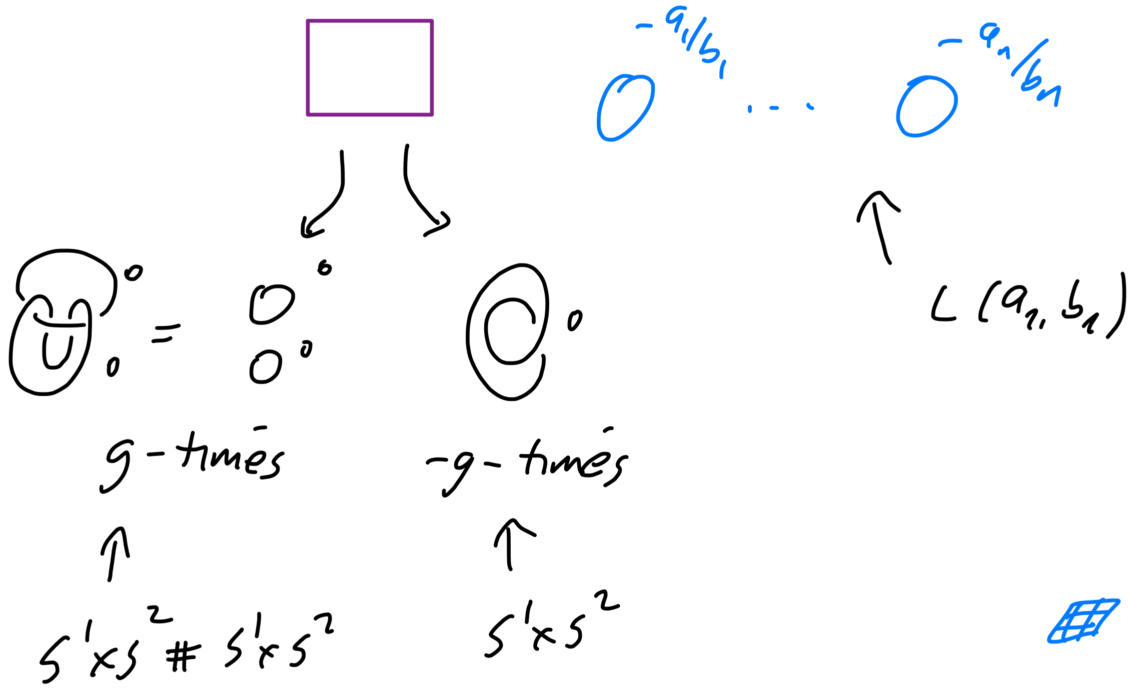
so can get to



now



is same as



$-a_1/b_1$
$-a_n/b_n$
0
0
=
0
0
0
g-times
-g-times
$S^1\times S^2 \# S^1\times S^2$
$S^1\times S^2$
$L(a_1, b_1)$